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LINEAR THEORY OF THE RAYLEIGH-TAYLOR INSTABILITY AT THE INTERFA--ETC(U)

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# Linear Theory of the Rayleigh-Taylor Instability at the Interface Between Two Compressible Media

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September 9, 1982

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# LINEAR THEORY OF THE RAYLEIGH-TAYLOR INSTABILITY AT THE INTERFACE BETWEEN TWO COMPRESSIBLE MEDIA

## 1. INTRODUCTION

The Rayleigh-Taylor instability occurs when a fluid supports a denser fluid against gravity, whereupon the two tend to interchange positions. It is encountered frequently in nature and in the laboratory. In inertial confinement fusion experiments, in which an ablatively driven medium implodes, compressing the material ahead of the ablation front to high densities, Rayleigh-Taylor instabilities can play a crucial role. They may occur in the ablative region, at the compression front, or (in the case of layered targets) at an interface between layers of different density.

The familiar treatments of the Rayleigh-Taylor instability which give rise to the classical dispersion relation (growth rate proportional to the square root of the product of perturbation wavenumber  $k$  and gravitational acceleration  $g$ ) presuppose that the fluid moves incompressibly, i.e., that the divergence of the flow field vanishes. It is frequently argued that when the effects of material compressibility are taken into account, the growth rates will be reduced in comparison with the incompressible case, since compression absorbs some of the energy which would otherwise go into fluid motion. One might speculate, for example, that growth rates at long wavelengths would be limited to a quantity of order the speed of sound divided by the wavelength. However, Bernstein et al.<sup>1</sup> have shown that in a broad class of general hydromagnetic systems the unstable modes with lowest threshold are associated with incompressible perturbations. It is thus conceivable that a compressible system might exhibit incompressible modes of interchange instability with the classical Rayleigh-Taylor growth rate.

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This is in fact the case for at least some models of imploding hydrodynamic systems.<sup>2,3</sup> To date no one seems to have suggested that compressibility might enhance the instability growth rates.

Previous attempts to determine the effect of compressibility on the Rayleigh-Taylor instability have tended to include numerous other effects as well,<sup>4</sup> obscuring the role played by compressibility, or have neglected to state clearly the assumptions used in the derivation,<sup>5,6</sup> leaving in doubt its domain of validity. The only conclusion these authors appear to have reached regarding compressibility is that its effects are small. In contrast, we have chosen to treat a model problem which can be solved exactly. We analyze the simple case of two contiguous semi-infinite slab regions in a constant gravitational field  $g$ , each filled with an ideal fluid whose density decreases exponentially in the upward direction (Fig. 1). If we subject this state to an infinitesimal perturbation, the resulting problem is free of such complicating features as thermal conduction, time dependence in the unperturbed state, curvilinear geometry, material flow across or along the interface, etc., and can be solved rigorously for the eigenfrequency  $\omega$  for either compressible (polytropic) or incompressible fluids.

When the media are incompressible we recover in the short-wavelength limit the expected result  $\omega^2 = -kg(\rho_0 - \rho_0')/(\rho_0 + \rho_0')$ , where  $\rho_0$  is the unperturbed density of the upper medium and  $\rho_0'$  is that of the lower, both measured at the interface. For longer wavelengths this simple form is replaced by one in which the respective scale heights  $p_0/\rho_0 g$  and  $p_0'/\rho_0' g$  enter, where  $p_0$  is the unperturbed pressure at the interface. In the limit  $k \rightarrow 0$ ,  $\omega$  becomes proportional to  $k$ . This wavenumber dependence reflects the structure of the basic state. In a (possibly more realistic) model involving two layers of finite thickness  $d, d'$ , the dispersion relation would naturally depend on the quantities  $kd, kd'$  and would be still more complicated.

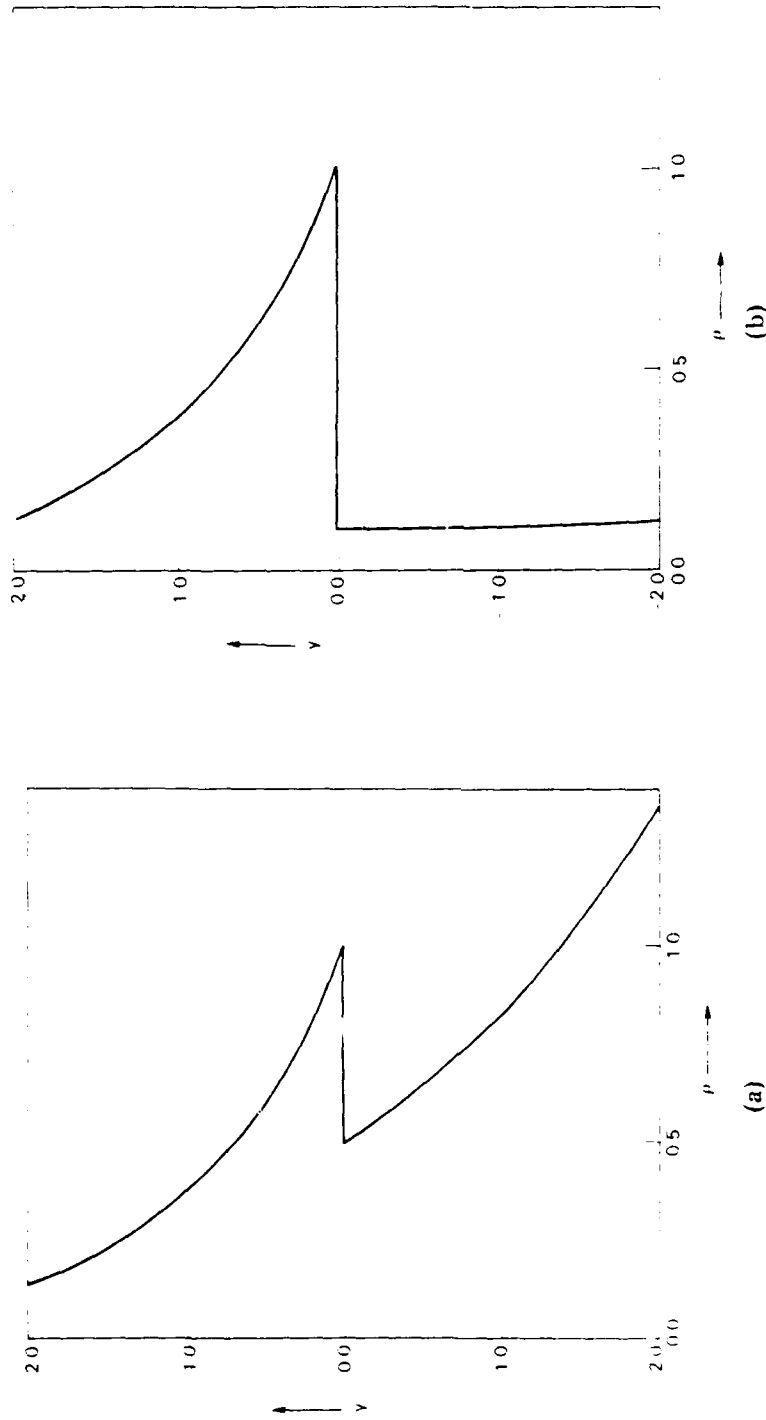


Fig. 1 — Density (horizontal axis) vs height for a system consisting of two constant-temperature media supported by pressure against gravity. The ratio of upper to lower density is (a) 2; (b) 10. The units are chosen to make the sound speed  $c$  in the upper region and the gravitational acceleration  $g$  both equal to unity.

Following the treatment of the incompressible case we turn to the compressible case. In the short-wavelength limit we once again obtain the classical result. This is unsurprising, as  $\omega^2/kg$  can depend on the sound speeds  $c, c'$  only through the dimensionless ratios  $kc^2/g, kc'^2/g$ . Large values of  $k$  are thus equivalent to large values of the sound speed, i.e., to incompressibility. When  $k$  is finite, however, we unexpectedly find that the growth rates are larger than the incompressible ones. We illustrate the result by solving the dispersion relation numerically as a function of wavenumber for two different values of the density jump at the interface and for several different values of  $\gamma$ , assuming that the latter has the same value in both regions. The difference in  $\omega^2/kg$  as  $\gamma$  varies between unity and  $\infty$  can amount to a factor  $\sim 2$ . These results, though they do not constitute an exhaustive parameter survey, typify what may be expected in real physical situations.

We conclude the paper with a brief discussion of our findings and some speculations on their significance.



## 2. INCOMPRESSIBLE FLUIDS

Consider a fluid described by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0 \quad (1)$$

and the momentum equation

$$\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p - \rho g \underline{e}_y, \quad (2)$$

where  $g$  is a constant. Thus  $-g \underline{e}_y$  is a uniform gravitational acceleration, oriented in the downward (negative  $y$ ) direction. These equations have an equilibrium solution

$$\underline{v} = 0; \quad (3)$$

$$p/\rho \equiv c^2 = \text{const}; \quad (4)$$

$$\nabla p = -\rho g \underline{e}_y = -\rho c^2 \frac{\partial}{\partial y}, \quad (5)$$

whence

$$\rho(y) = \rho_0 \exp(-gy/c^2), \quad (6)$$

$\rho_0$  constant. We take the interface separating the two media in the problem to be located at  $y = 0$ , i.e., coinciding with the  $x$ -axis (see Fig. 1). To distinguish the two regions we will label all quantities belonging to the lower one with primes. Mechanical equilibrium requires pressure balance,

$$p(0) = p'(0), \quad (7)$$

but in general  $\rho_0 \neq \rho'_0$  and  $c \neq c'$ . Since we anticipate that instability will occur when the upper medium is denser (at  $y = 0$ ) than the lower, we assume  $c' > c$ .

Suppose that this state is subjected to an infinitesimal perturbation, defined by the local displacement  $\xi(x, y, t)$  of an element of fluid.

Using the subscript "1" to distinguish perturbed quantities, we have for the velocity

$$\chi_1 = \frac{\partial \xi}{\partial t}, \quad (8)$$

so that the perturbed density satisfies

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot \rho \chi_1 = -\frac{\partial}{\partial t} \nabla \cdot \rho \xi, \quad (9)$$

whence

$$\rho_1 = -\nabla \cdot \rho \xi. \quad (10)$$

The incompressibility condition

$$\nabla \cdot \chi = 0 \quad (11)$$

implies

$$\nabla \cdot \xi = 0 \quad (12)$$

as well, so that Eq. (10) becomes simply

$$\rho_1 = -\xi \cdot \nabla \rho = \rho_0 \xi_y / c^2 \quad (13)$$

by virtue of Eq. (5). Thus the linearized momentum equation becomes

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -\nabla p_1 - \frac{\partial}{\partial y} \rho_1 = -\nabla p_1 - \frac{\rho g^2 \xi_y}{c^2} \frac{\partial}{\partial y}. \quad (14)$$

The y-component of Eq. (14) yields

$$\rho \frac{\partial^2 \xi_y}{\partial t^2} = -\frac{\partial p_1}{\partial y} - \frac{\rho g^2 \xi_y}{c^2}. \quad (15)$$

Dividing Eq. (14) by  $\rho$  and taking the divergence, we find using Eq. (12) that

$$0 = -\nabla \cdot \left( \frac{1}{\rho} \nabla p_1 \right) - \frac{\rho g^2}{c^2} \frac{\partial \xi_y}{\partial y}. \quad (16)$$

Setting

$$p_1 = p \lambda = \rho c^2 \lambda, \quad (17)$$

we can rewrite Eqs. (15) and (16) as

$$\frac{\partial^2 \xi_y}{\partial t^2} = -c^2 \frac{\partial \lambda}{\partial y} + g \lambda - \frac{g^2 \xi_y}{c^2} \quad (18)$$

and

$$0 = c^2 \nabla^2 \lambda + g \frac{\partial \lambda}{\partial y} - \frac{\pi^2}{c^2} \frac{\partial \xi_y}{\partial x}. \quad (19)$$

Let us seek solutions of the form

$$\xi_y = A \exp(ikx - \mu y - i\omega t), \quad (20)$$

$$\lambda = B \exp(ikx - \mu y - i\omega t). \quad (21)$$

Equations (20)-(21) include a factor  $\exp(-i\omega t)$ , which may be either oscillatory or exponential, depending on  $\omega$ . Its presence implies that we are seeking ordinary (nonsingular) normal modes of the system. This restriction is equivalent to solving the initial value problem associated with Eq. (14), e.g., by carrying out a Laplace transform, and then throwing out the portion of the solution which depends on the initial data. We return to this point in our concluding discussion (Section 4).

Substitution of Eqs. (20)-(21) in Eqs. (18)-(19) yields

$$-\omega^2 A = \mu c^2 B + gB - (g^2/c^2)A, \quad (22)$$

$$0 = -c^2(\mu^2 - k^2)B - g\mu B + (g^2/c^2)\mu A. \quad (23)$$

Thus  $\mu$  can be found from the solubility condition, i.e., by eliminating  $A/B$  between

$$\frac{A}{B} = \frac{g\mu - c^2(k^2 - \mu^2)}{\mu g^2/c^2} \quad (24)$$

and

$$\frac{A}{B} = \frac{\mu c^2 + g}{g/c^2 - \omega^2}. \quad (25)$$

Solving the resulting quadratic equation, we find in the upper half plane, taking the square root with positive real part,

$$\mu = \frac{-z + (4k^2c'^4 + g^2 - 4g^2k^2c'^2/\omega^2)^{1/2}}{2c'^2} \quad (26)$$

in order that Eqs. (20) and (21) not diverge, while in the lower half plane, where  $\text{Re} \mu' < 0$  is required,

$$\mu' = \frac{-z - (4k^2c'^4 + g^2 - 4g^2k^2c'^2/\omega^2)^{1/2}}{2c'^2}, \quad (27)$$

at least for  $\omega^2 < 0$ .

The kinematic boundary condition on the interface reduces to continuity of the vertical displacement  $\xi_y$ , i.e.,

$$A = A'. \quad (28)$$

The dynamic boundary condition requires that the pressure be continuous. The perturbed pressure on the displaced boundary is

$$\begin{aligned} [p_1 + \xi \cdot \nabla p]_{y=0} &= [p_1 - \xi_y p z / c^2]_{y=0} \\ &= [\rho p - \xi_y \rho g / c^2]_{y=0} \\ &= (B - gA/c^2)p(0). \end{aligned} \quad (29)$$

Continuity of the unperturbed pressure [Eq. (7)] thus implies

$$B - gA/c^2 = B' - gA'/c'^2. \quad (30)$$

Together Eqs. (28) and (30) yield

$$\frac{B}{A} - \frac{g}{c^2} = \frac{B'}{A'} - \frac{g}{c'^2}. \quad (31)$$

Substituting Eq. (24) or (25) and its primed counterpart in this expression and employing Eqs. (26) and (27), we obtain an equation for the eigenfrequency  $\omega$ .

Using the interactive symbolic manipulation system MACSYMA, we transformed this equation into an algebraic equation in  $z = \omega^2/g$ . This was done by squaring the equation twice to eliminate the square roots arising from Eqs. (26) and (27). Ignoring the spurious roots this procedure introduces, we are left with

$$\pm = \frac{1 \pm (1 + D^2)^{1/2}}{D}, \quad (32)$$

where

$$D = k(p^2 - p'^2)g^{-1} \quad (33)$$

and

$$E = k(p^2 + p'^2)g^{-1} \quad (34)$$

In Eq. (32), the upper (lower) sign is associated with positive (negative) values of  $D$ . When we take  $kp^2 \gg g$ ,  $kp'^2 \gg g$  (wavelength short compared with both scale heights), we recover the usual dispersion relation for the Rayleigh-Taylor instability at an interface between two uniform media, viz.,

$$-\frac{\omega^2}{k_g} = \frac{\rho_0 - \rho_0'}{\rho_0 + \rho_0'}. \quad (35)$$

At long wavelengths ( $k \rightarrow 0$ ), Eq. (32) goes over to

$$\omega^2 = -k^2 \frac{(p^2 - p'^2)^2}{2(p^2 + p'^2)}, \quad (36)$$

displaying the effect of the spatial dependence of the unperturbed state.

The quantity  $\omega^2/k_g$  can depend on  $p$  and  $p'$  only through the dimensionless parameters  $kp^2/g = k\rho(p)/\rho_0 g$  and  $kp'^2/g = k\rho'(p')/\rho_0' g$ . Assuming that the wavelength is long only in comparison with the upper scale height is equivalent to letting the density of the upper medium go to infinity, whereupon Eq. (32) becomes

$$\frac{-\omega^2}{k_g} = \frac{1 - (1 + k^2 p'^2 / p^2)^{1/2}}{k p'^2 / p^2}. \quad (37)$$

## 5. COMPRESSIBLE MEDIA

Here we replace the condition (31) with

$$\frac{\partial p}{\partial t} + \gamma \cdot \nabla p + \gamma p \nabla \cdot \gamma = 0, \quad (38')$$

i.e., we assume the fluid is a polytrope with ratio of specific heats  $\gamma$ . The linearized form of Eq. (38'),

$$\frac{\partial p_1}{\partial t} + \gamma_1 \cdot \nabla p_1 + \gamma p \nabla \cdot \gamma_1 = 0, \quad (39')$$

is solved analogously to Eq. (3) to obtain

$$p_1 = -\gamma p \nabla \cdot \xi - \xi \cdot \nabla p. \quad (40')$$

Substitution of Eqs. (10') and (40') in the first-order momentum equation yields

$$\begin{aligned} \rho \frac{\partial^2 \xi}{\partial t^2} &= \nabla (\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) + \rho \xi \nabla \cdot \nabla p \\ &= \nabla (\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) + \rho \xi \nabla \cdot \nabla p \\ &= \rho \frac{\partial p}{\partial t} (\gamma \nabla \cdot \xi + \xi \cdot \nabla \gamma) + \rho \nabla (\gamma \nabla \cdot \xi + \xi \cdot \nabla \gamma) \\ &\quad - \rho \xi \nabla \cdot \nabla p, \end{aligned} \quad (41')$$

or

$$\frac{\partial^2 \xi}{\partial t^2} = -(\gamma-1) \xi \nabla \cdot \nabla p + \nabla (\gamma \nabla \cdot \xi + \xi \cdot \nabla \gamma), \quad (42')$$

where we have defined

$$\sigma = \gamma \nabla \cdot \xi. \quad (43')$$

Note that  $c^2$  and  $c'^2$  differ from the squares of the respective sound speeds by factors of  $\gamma$  and  $\gamma'$ . Taking the divergence of Eq. (42') results in an equation for  $\sigma$ :

$$\frac{\partial^2 \sigma}{\partial t^2} = -(\gamma-1) \xi \nabla \cdot \nabla p + \nabla^2 (\gamma \nabla \cdot \xi + \xi \cdot \nabla \gamma), \quad (44')$$

while the  $y$ -component of Eq. (42') yields

$$\frac{\partial^2 \xi_y}{\partial t^2} = -(\gamma-1) \xi \sigma + \frac{\partial}{\partial y} (\gamma \nabla \cdot \xi + \xi \cdot \nabla \gamma). \quad (45')$$

Once again we seek solutions in the form of exponential functions of the form of exponential functions of the independent variables, using Eq. (20) and

$$\sigma(x, y, t) = C \exp(ikx - \mu y - i\omega t). \quad (46)$$

Equations (44) and (45) imply

$$-\omega^2 A = -(\gamma - 1)gC - \mu(\gamma c^2 C - gA), \quad (47)$$

and

$$-\omega^2 C = \mu(\gamma - 1)gC + (\mu^2 - k^2)(\gamma c^2 C - gA), \quad (48)$$

whence

$$\frac{A}{C} = - \frac{\omega^2 + (\gamma - 1)\mu g - (k^2 - \mu^2)\gamma c^2}{(k^2 - \mu^2)g} = \frac{(\gamma - 1)g + \gamma c^2 \mu}{\omega^2 + \mu g}. \quad (49)$$

The solutions in the upper and lower half-planes are respectively

$$\begin{aligned} \mu = & [-\gamma g + (\gamma^2 g^2 - 4\omega^2 \gamma c^2 + 4\gamma^2 k^2 c^4 - \\ & 4\gamma(\gamma - 1)g^2 k^2 c^2 / \omega^2)^{1/2}] (2\gamma c^2)^{-1} \end{aligned} \quad (50)$$

and

$$\begin{aligned} \mu' = & [-\gamma' g - (\gamma'^2 g^2 - 4\omega^2 \gamma' c'^2 + 4\gamma'^2 k^2 c'^4 - \\ & 4\gamma'(\gamma' - 1)g^2 k^2 c'^2 / \omega^2)^{1/2}] (2\gamma' c'^2)^{-1}. \end{aligned} \quad (51)$$

Pressure balance requires that  $p_1 + \frac{1}{2} \rho p$  be continuous at  $y = 0$ , whence by Eq. (40)

$$\gamma p(0)\sigma = \gamma' p'(0)\sigma', \quad (52)$$

or by virtue of Eq. (7),

$$\gamma C = \gamma' C'. \quad (53)$$

Combining Eqs. (28) and (53) we have

$$\frac{A}{\gamma C} = \frac{A'}{\gamma' C'}. \quad (54)$$

By substituting Eqs. (50) and (51) in Eq. (49) and its primed counterpart, respectively, we obtain from Eq. (54) a relation determining  $\omega$ . Again using MAJIMA, we follow a procedure formally identical to that used in deriving

Eq. (32). The physical root is found to satisfy the quartic

$$\tilde{D}^2 Z^4 - 2D\tilde{D}SZ^3 + (D^2S^2 + 2D\tilde{D} - \tilde{D}^2)Z^2 - 2D^2(S - \tilde{S})Z - D^4 = 0, \quad (55)$$

where  $D$  and  $S$  are defined in Eqs. (33)-(34) and

$$\tilde{D} = k(c^2/\gamma - c'^2/\gamma')g^{-1}, \quad (56)$$

$$\tilde{S} = k(c^2/\gamma + c'^2/\gamma')g^{-1}. \quad (57)$$

Evidently Eq. (55) always has one negative root. This can be exhibited by applying the general procedure for solving a quartic, but the result is far too cumbersome to be useful. Instead we look at some limits and special cases of physical interest.

First let  $\gamma' = \gamma$ , so that both regions contain fluids with the same compressibility properties. Then Eq. (55) becomes

$$Z^4 - 2\gamma SZ^3 + (\gamma^2 S^2 + 2\gamma - 1)Z^2 - 2\gamma(\gamma - 1)SZ - \gamma^2 = 0. \quad (58)$$

In the limit  $\gamma \rightarrow \infty$ , the solution of Eq. (58) associated with the instability satisfies

$$S^2 Z^2 - SZ - D^2 = 0, \quad (59)$$

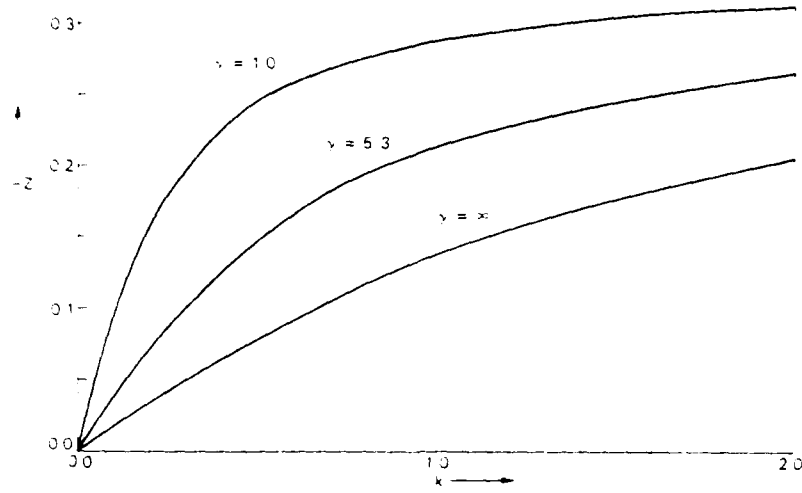
whose negative root is given by Eq. (32). We thus recover the incompressible result, as expected. Figure 2 illustrates the approach to this limit. It shows plots of  $-\omega^2/kg$  as functions of  $k$  obtained by solving Eq. (54) or (55) numerically for various choices of  $\gamma$  between 1 and  $\infty$ , assuming the unperturbed states shown in Fig. 1. As can be seen, finite compressibility increase the growth rates, the relative effect being greatest at long wavelengths. When  $k \rightarrow 0$  then  $S$ ,  $D$ ,  $\tilde{S}$  and  $\tilde{D}$  all become small and Eq. (55) reduces for finite  $\gamma$ ,  $\gamma'$  to

$$\tilde{D}(2D - \tilde{D})Z^2 - 2D^2(S - \tilde{S})Z - D^4 = 0, \quad (60)$$

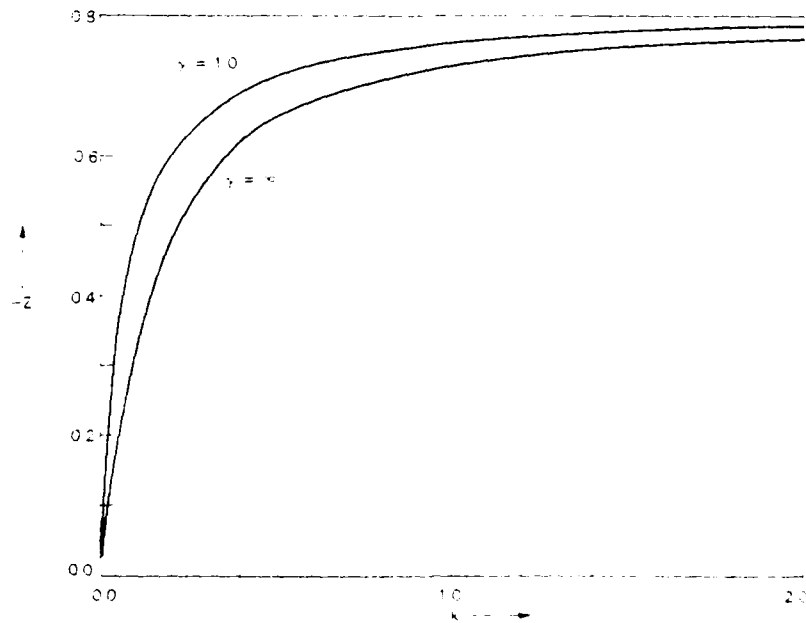
whence  $\omega$  is proportional to  $k$ . For  $\gamma' = \gamma$  Eq. (60) yields

$$Z = \frac{\gamma}{2\gamma - 1} \{ (\gamma - 1)S - [(\gamma - 1)^2 + (2\gamma - 1)D^2]^{1/2} \}. \quad (61)$$





(a)



(b)

Fig. 2 — Dimensionless squared growth rate  $-\omega^2/kg$  vs wavenumber  $k$  for the two basic states shown in Fig. 1(a), (b), with the same choice of units as in the latter. The adiabatic index  $\gamma$  in both regions is taken to be 1, 5/3 or  $\infty$ , as indicated by the label. Note that the curves asymptotically approach the value  $(\rho - \rho')/(\rho + \rho')$ , equal to 0.333 and 0.818, respectively.

When  $\gamma = 1$ , this becomes

$$\omega^2 = k^2(c^2 - c'^2). \quad (62)$$

This is to be compared with the corresponding incompressible result given in Eq. (36). On the other hand, for short-wavelength perturbations ( $k \rightarrow \infty$ ), Eq. (55) reduces to

$$g^2 z^2 - p^2 = 0, \quad (63)$$

whose solution is identical with Eq. (35).

Another interesting case is that in which the density of the upper medium becomes infinite, so that  $c = 0$ . One of the extraneous roots factors out of Eq. (55), which then reduces to a cubic,

$$z^3 + (2\gamma' - 1)z - \frac{2\gamma'c'^2}{g} (z^2 - 1) = 0. \quad (64)$$

Equation (64) holds even if  $\gamma \rightarrow \infty$  in such a way that  $\gamma c'^2$  (the square of the sound speed) remains finite. If instead we assume that the density in the lower region vanishes (i.e.,  $c'^2 \rightarrow \infty$ ), then the dispersion relation is even simpler, becoming

$$z = -1, \quad (65)$$

which coincides with the incompressible result.

Finally, if we assume

$$\rho_0 \gamma = \rho_0' \gamma', \quad (66)$$

then  $\tilde{D}$  vanishes identically and Eq. (55) simplifies to

$$g^2 z^2 - g(z - \tilde{S})^2 - p^2 = 0, \quad (67)$$

whence

$$z = \frac{g - \tilde{S} \pm [(g - \tilde{S})^2 + p^2]^{1/2}}{g}. \quad (68)$$

If

$$\rho_0 = \rho_0', \quad (69)$$

so that  $c^2 = c'^2$  and  $D = 0$ , then

$$z = 1, \quad (70)$$

i.e., the perturbations are marginally stable.

#### 4. CONCLUSIONS

We have shown that, for the particular model investigated here, the growth rate of the Rayleigh-Taylor instability decreases with increasing  $\gamma$ . Compressible systems are all more unstable than the incompressible one, although the growth rates are smaller than those obtained in the case of uniform incompressible media [Eq. (35)]. This does not contradict the familiar energy-principle argument (Section 1) that incompressible perturbations have thresholds as low as or lower than compressible ones, because in all cases discussed here we have the same stability criterion, namely,  $\rho_0 \leq \rho_0'$ . Nor is there any paradox associated with the finite speed of sound, the velocity with which disturbances propagate in compressible media. In this treatment we have considered the evolution only of eigenmodes, which by definition are initiated in "prepared" states involving the entire system,  $-\infty < x, y < \infty$ . We could use Laplace transforms in time to solve the initial value problem associated with the evolution of an arbitrary perturbation with horizontal wavenumber  $k$ . We would then find poles in the transform corresponding to the roots of Eq. (55) [Eq. (32) in the compressible case]. In addition the transform would exhibit branch points on the real axis corresponding to a continuous spectrum of eigenvalues. These give rise to a continuum of modes which are necessary to solve the problem with arbitrary initial conditions, but which lead to decaying solutions and hence need not be considered if one is concerned only with stability. In the compressible case the initial data determine a superposition of these continuum modes. The resulting signal propagates away from  $y=0$  into the rest of the system with the speed of sound.

Evidently the onset of the process by which the two participating fluids interchange positions is aided by the extra degrees of freedom which allow them to compress and expand. A mechanical analogy is supplied by the example of the inverted physical pendulum (e.g., a brick smokestack), which falls over upon undergoing an angular displacement. Different parts of the pendulum falling individually would assume different angular velocities due to the action of gravity alone. Hence the pendulum tends to bend or break up under internal stresses, allowing it to fall over faster than one constrained to remain rigid.

Since the range of effective values of  $\gamma$  for materials of interest in inertial confinement fusion is not great, the quantitative changes in instability growth rates (vis-à-vis those found in incompressible theory) are likely to be modest, perhaps a few tens of percent. Furthermore, the present calculation is admittedly idealized, and the results are certainly model-dependent. Nevertheless, our findings do not encourage one to anticipate that compressibility effects can mitigate the Rayleigh-Taylor instability. Perhaps the only conclusion should be that the effects of compressibility need to be carefully treated in each individual situation.

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